Two-dimensional cylindric and toroidal codes

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Decision problem

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- a **yes-or-no** question

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Examples:

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Examples:

- "how many prime numbers are less than 100?"

  not a decision problem

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  not a decision problem
- "is 7 prime?" **not** a decision problem

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- a **yes-or-no** question
- posed on a specified **set of inputs**

Examples:

- “how many prime numbers are less than 100?” **not** a decision problem
- “is 7 prime?” **not** a decision problem
- “is x prime?” a decision problem

Decidability

Does exists a **decision procedure** (an algorithm) that solves the problem?

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(a) yes

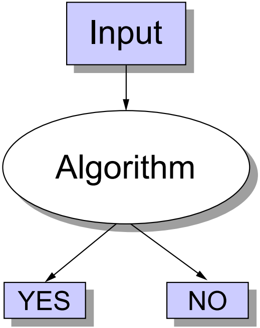

Figure from Wikipedia

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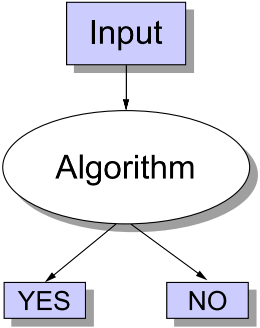

Figure from Wikipedia

(b) no

Halting problem

Given

- a Turing machine M (or equivalently, an algorithm)
- and input x

does the computation of M on x halt or not?

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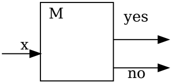
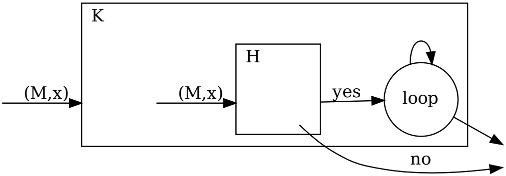
does the computation of M on x halt or not?

Spoiler: it is undecidable (Alan Turing, 1936).

Undecidability of the halting problem



Let $H(M, x)$ be a T.M. that solves the halting problem.



Strings and string codes

- string: aab
- factorization of aab $\{a, ab\}$: a, ab
- code of strings: a set of strings such that each string has at most one factorization

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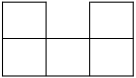
Example

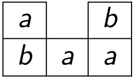
$\{a, b, ab\}$ is not a set:

$$aaba \implies (a, ab, a), (a, a, b, a)$$

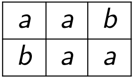
In two dimensions



(a) Polyominoes



(b) Bricks

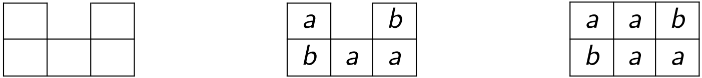


(c) Pictures

[1]Figures by M. Madonia

[2]Figures from J. Sullivan, *Conformal Tiling on a Torus*

In two dimensions



(a) Polyominoes (b) Bricks (c) Pictures

- codes of pictures

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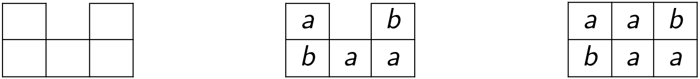

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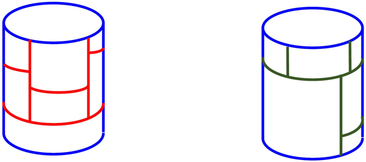
- cylindric codes of pictures [1]



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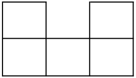

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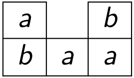

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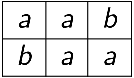

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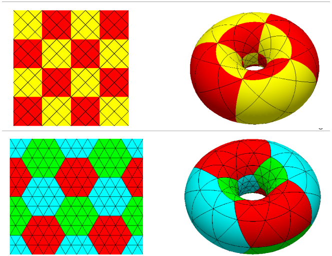

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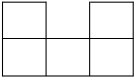

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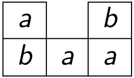

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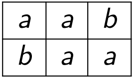

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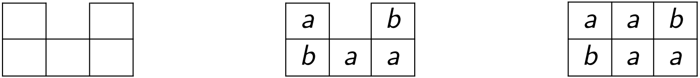

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All undecidable problems!

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Reductions

A reduction from a problem A to a problem B is a function

- from the set of inputs of A
- to the set of inputs of B

such that

$$A(a) = \text{yes} \iff B(f(a)) = \text{yes}$$
$$A(a) = \text{no} \iff B(f(a)) = \text{no}$$

Reductions and undecidability

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function $\mathrm{F}(a)$

$\quad\dots$

Reductions and undecidability

function $\text{F}(a)$

$\ldots$

procedure $\textsc{solve}\text{B}(b)$

$\ldots$

Reductions and undecidability

function $\textsc{f}(a)$
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procedure $\textsc{solveB}(b)$
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procedure $\textsc{solveA}(a)$
    $b \leftarrow \textsc{f}(a)$
    return $\textsc{solveB}(b)$

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B decidable $\implies$ A decidable

Reductions and undecidability

function $\textsc{f}(a)$

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$\quad \cdots$

procedure $\textsc{solveA}(a)$

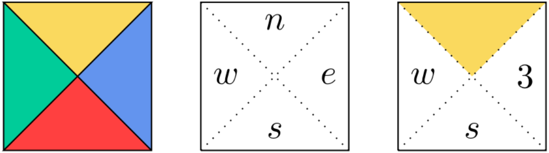
$\quad b \leftarrow \textsc{f}(a)$

$\quad$ **return** $\textsc{solveB}(b)$

$$B \text{ decidable} \implies A \text{ decidable}$$

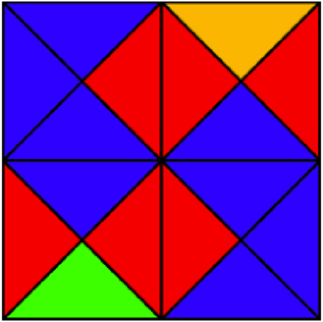
$$A \text{ undecidable} \implies B \text{ undecidable}$$

Wang tiles

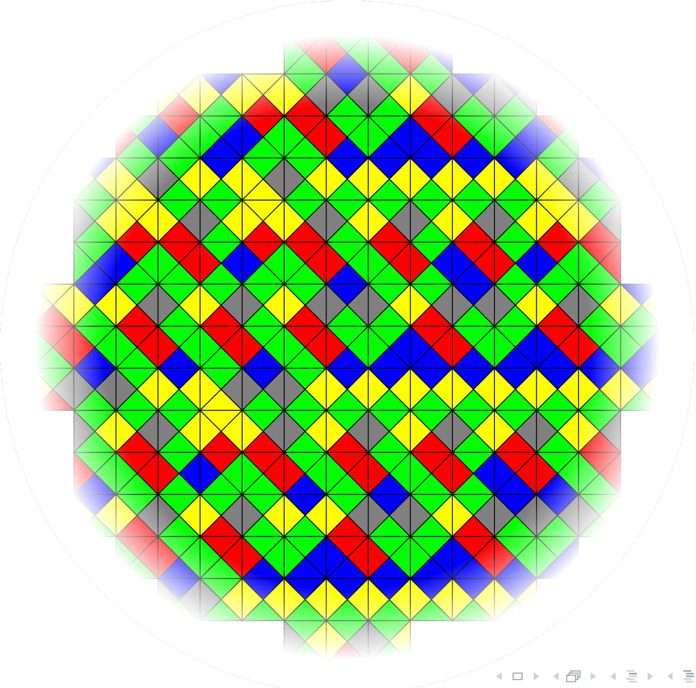


[0] figure from Thijs van Loenhout, *Wang Tiles and Cubes*

Wang tiles adjacency condition



[0] figure from Toshiaki Matsushima, Yoshihiro Mizoguchi, and Alexandre Derouet-Jourdan, *Verification of a brick Wang tiling algorithm*



Undecidable problems on Wang tiles

Given a set of Wang tiles

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- does exists a tiling of the plane?

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Reduction used in the proof I

We construct a reduction

- from the **Wang periodic tiling problem** (A)
- to the **toroidal codicity problem** (B)

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$$A \text{ undecidable} \implies B \text{ undecidable}$$

Reduction used in the proof II

$$f : X \mapsto X_W$$

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- $\begin{array}{|ccc|}\hline & b & \\ a & & c \\ & d & \\ \hline\end{array} \implies \begin{array}{|ccc|}\hline \star & \hat{b} & \star \\ \acute{a} & \star & \grave{c} \\ \star & \check{d} & \star \\ \hline\end{array}$

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- $a \in \Sigma \Longrightarrow \boxed{\grave{a} \quad \acute{a}}, \boxed{\begin{array}{c} \check{a} \\ \hat{a} \end{array}}$
- $\boxed{\star}$

Is this function a reduction, i.e. $A(a) = B(f(a))$?

Reduction used in the proof II

$$f : X \mapsto X_W$$

- $\boxed{\begin{matrix} & b & \\ a & & c \\ & d & \end{matrix}} \implies \boxed{\begin{matrix} \star & \hat{b} & \star \\ \acute{a} & \star & \grave{c} \\ \star & \check{d} & \star \end{matrix}}$
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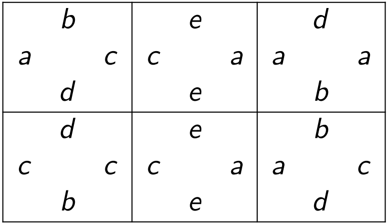
- $\boxed{\star}$

Is this function a reduction, i.e. $A(a) = B(f(a))$?
Yes!

$$W \text{ has a periodic tiling } \iff X_W \text{ is not a toroidal code}$$

W has a periodic tiling $\implies X_W$ is not a toroidal code

Given a periodic tiling, we can build a picture with two toroidal decomposition:



(a) Periodic tiling

$$
\begin{array}{|ccccccccc|}
\hline
\star & \hat{b} & \star & \star & \hat{e} & \star & \star & \hat{d} & \star \\
\acute{a} & \star & \grave{c} & \acute{c} & \star & \grave{a} & \acute{a} & \star & \grave{a} \\
\star & \check{d} & \star & \star & \check{e} & \star & \star & \check{b} & \star \\
\star & \hat{d} & \star & \star & \hat{e} & \star & \star & \hat{b} & \star \\
\acute{c} & \star & \grave{c} & \acute{c} & \star & \grave{a} & \acute{a} & \star & \grave{c} \\
\star & \check{b} & \star & \star & \check{e} & \star & \star & \check{d} & \star \\
\hline
\end{array}
$$

(b) Associated picture

W has a periodic tiling $\implies$ X_W is not a toroidal code

$\star$	$\hat{b}$	$\star$	$\star$	$\hat{e}$	$\star$	$\star$	$\hat{d}$	$\star$
$\acute{a}$	$\star$	$\grave{c}$	$\acute{c}$	$\star$	$\grave{a}$	$\acute{a}$	$\star$	$\grave{a}$
$\star$	$\check{d}$	$\star$	$\star$	$\check{e}$	$\star$	$\star$	$\check{b}$	$\star$
$\star$	$\hat{d}$	$\star$	$\star$	$\hat{e}$	$\star$	$\star$	$\hat{b}$	$\star$
$\acute{c}$	$\star$	$\grave{c}$	$\acute{c}$	$\star$	$\grave{a}$	$\acute{a}$	$\star$	$\grave{c}$
$\star$	$\check{b}$	$\star$	$\star$	$\check{e}$	$\star$	$\star$	$\check{d}$	$\star$

(a) Fist decomposition

$\star$	$\hat{b}$	$\star$	$\star$	$\hat{e}$	$\star$	$\star$	$\hat{d}$	$\star$
$\acute{a}$	$\star$	$\grave{c}$	$\acute{c}$	$\star$	$\grave{a}$	$\acute{a}$	$\star$	$\grave{a}$
$\star$	$\check{d}$	$\star$	$\star$	$\check{e}$	$\star$	$\star$	$\check{b}$	$\star$
$\star$	$\hat{d}$	$\star$	$\star$	$\hat{e}$	$\star$	$\star$	$\hat{b}$	$\star$
$\acute{c}$	$\star$	$\grave{c}$	$\acute{c}$	$\star$	$\grave{a}$	$\acute{a}$	$\star$	$\grave{c}$
$\star$	$\check{b}$	$\star$	$\star$	$\check{e}$	$\star$	$\star$	$\check{d}$	$\star$

(b) Second decomposition

X_W is not a toroidal code $\implies$ W has a periodic tiling

There exists a picture with toroidal tilings:

- such picture cannot contain only $\star$
- there must be a letter which is covered differently in the two tilings

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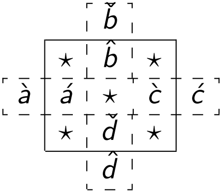

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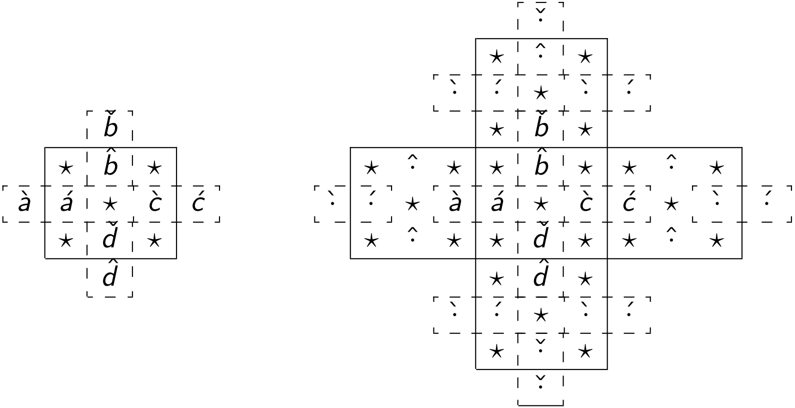
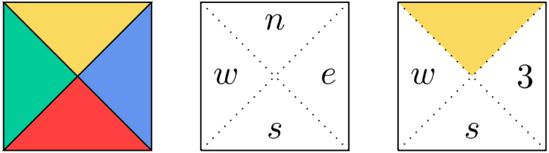


Thanks for your attention!

Wang tiles



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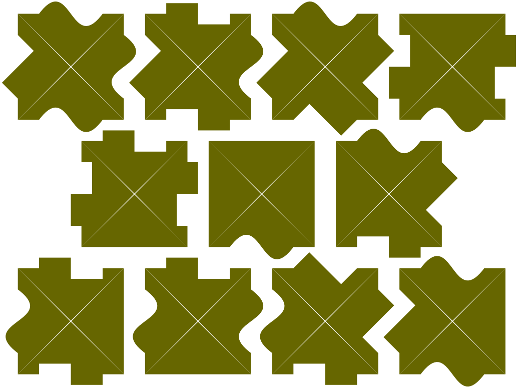
 [2]

[2] figure from Wikipedia

Decision problem

$$\mathcal{P} \colon A \to \{0, 1\}$$

Examples:

- "is x prime?"
- "is a propositional/first order/... formula ϕ satisfiable?"

Decidability

$\mathcal{P}$ is decidable if exists a Turing machine/algorithm $\mathcal{M}$ such that

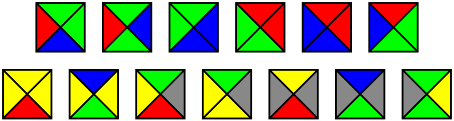
$$\mathcal{M}(a) = \mathcal{P}(a) \qquad \text{for all } a \in A$$

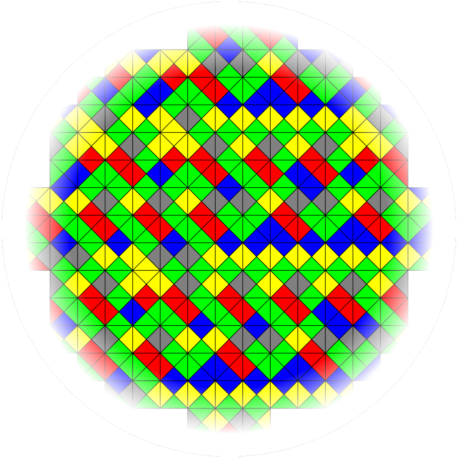
Decision problems on Wang tiles

Given a set of Wang tiles, does exists...

- a tiling of the plane?
- a periodic tiling of the plane?
- an aperiodic tiling of the plane?

A minimal aperiodic tiling set





[3]

[3] figures from Wikipedia

How to prove decidability?

By constructing an algorithm that solves the problem! e.g.:

procedure $\textsc{Sieve of Eratosthenes}(n)$
    for $i \leftarrow 2, 3, \ldots, \lfloor \sqrt{n} \rfloor$ **do**
        if $n \mod i = 0$ **then**
            return 1
    return 0

How to prove undecidability?

Usually one of two ways:

- by contradiction (*reductio ad absurdum*)
- by a reduction from another undecidable problem

Reductions

Given

$$\mathcal{A}\colon A \to \{0,1\}$$
$$\mathcal{B}\colon B \to \{0,1\}$$

a reduction from $\mathcal{A}$ to $\mathcal{B}$ is a function

$$f : A \to B$$

such that

$$\mathcal{B}(f(a)) = \mathcal{A}(a) \qquad \text{for all } a \in A$$

f must be

- total: $\operatorname{dom} f = A$
- computable: exists a Turing machine/algorithm that computes f

Why reductions?

$$f\colon A \to B \qquad \mathcal{B}(f(a)) = \mathcal{A}(a) \qquad f \text{ computable}$$

$$\mathcal{B} \text{ decidable} \implies \mathcal{A} \text{ decidable}$$
$$\mathcal{A} \text{ undecidable} \implies \mathcal{B} \text{ undecidable}$$

Reductions from and to problems on Wang tiles

$$\mathcal{A} \leq \mathcal{W} \leq \mathcal{B}$$

Which $\mathcal{A}$ and $\mathcal{B}$?

- proof of their undecidability: reduction the from halting problem
- useful to prove the undecidability of many problems; the first of these results is the undecidability of $\forall\exists\forall$ fragment of first order logic (Wang)

formulae of $\forall\exists\forall$ fragment:

$$(\forall \ldots \forall)(\exists \ldots \exists)(\forall \ldots \forall)\phi, \qquad \phi \text{ propositional formula}$$

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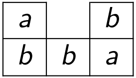
formulae of $\forall\exists\forall$ fragment:

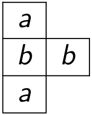
$$(\forall \dots \forall)(\exists \dots \exists)(\forall \dots \forall)\phi, \qquad \phi \text{ propositional formula}$$

Undecidability of brick codes I

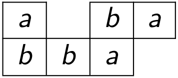
- bricks:

$k_1 =$ 

$k_2 =$ 

$k_3 =$ 

- a brick tilable with k_1 and k_2:



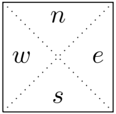
- a set of bricks X is a code if every brick has **at most one** tiling with the elements of X

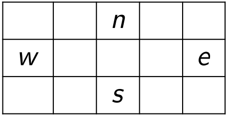
Undecidability of brick codes II

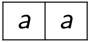
Undecidability provable by reduction from finite Wang tiling problem:

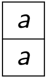
- a tile can have some edges (but not all) labelled with a blank colour
- does exists a finite tiling with blank border?

Undecidability of brick codes III

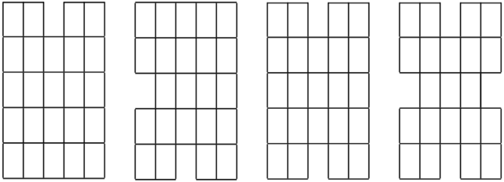
- for each tile 

: 

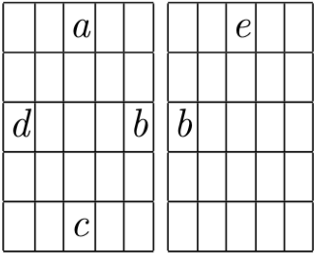
- for each colour a: 

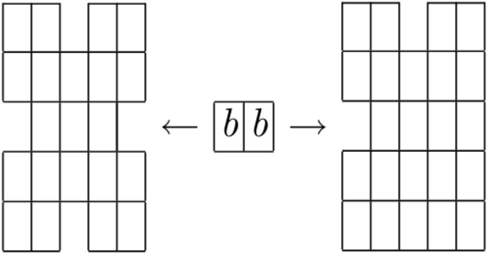


- all the blank shapes with one or more edge mid-points removed, e.g.:



Undecidability of brick codes IV





Thanks for your attention!